

Derivation of mean-field equations for stochastic particle systems

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Abstract

We study the single site dynamics in stochastic particle systems of misanthrope type with bounded rates on a complete graph. In the limit of diverging system size we establish convergence to a Markovian non-linear birth death chain, described by a mean-field equation known also from exchange-driven growth processes. Conservation of mass in the particle system leads to conservation of the first moment for the limit dynamics, and to non-uniqueness of stationary measures. The proof is based on a coupling to branching processes via the graphical construction, and establishing uniqueness of the solution for the limit dynamics. As particularly interesting examples we discuss the dynamics of two models that exhibit a condensation transition and their connection to exchange-driven growth processes.

Keywords. mean-field equations, misanthrope processes, non-linear birth death chain, condensation.

1 Introduction

In the physics literature, stochastic particle systems in a limit of large system size are often described by a mean-field master equation for the time evolution of a single lattice site [1, 2, 3]. For conservative systems, these equations are very similar to mean-field rate equations in the study of cluster growth models (see e.g. [4] and the references therein). We focus on particle systems where only one particle jumps at a time, which corresponds to monomer exchange in cluster growth models as studied in [5], and also in the well-known Becker-Döring model [6, 7]. While these mean-field equations often provide the starting point for the analysis and have an intuitive form, to our knowledge their connection to underlying particle systems has not been rigorously established so far.

In this paper, we provide a rigorous derivation of this equation for misanthrope-type processes [8] with bounded jump rates and homogeneous initial conditions on a complete graph. The limit equation describes the dynamics of the fraction $f_k(t) \in [0, 1]$ of lattice sites with a given occupation number k , and also provides the master equation of a birth death chain for the limiting single site dynamics of the process. Note that no time rescaling is required and the limiting dynamics are non-linear, i.e. the birth and death rates of the chain depend on the distribution $f_k(t)$. Even

though the limiting birth death dynamics is irreducible under non-degenerate initial conditions, the non-linearity leads to conservation of the first moment of the initial distribution, resulting in a continuous family of stationary distributions, as has been observed before for other non-linear birth death chains (see e.g. [9]). To establish the mean-field property in the limit, we show the asymptotic decay of correlations by bounding percolation clusters in the graphical construction of the process with branching processes up to finite times, similar to [10, 11]. Existence of limits follows from standard tightness arguments, and our proof also includes a simple uniqueness argument for solutions of the limit equation. While uniqueness has been established for more complicated coagulation fragmentation models [7], we could not find a result covering our case in the literature.

Under certain conditions on the jump rates, stochastic particle systems can exhibit a condensation transition where a non-zero fraction of all particles accumulates in a condensate, provided the particle density exceeds a critical value ρ_c . Condensing models with homogeneous stationary product measures have attracted significant research interest (see e.g. [12, 3] for recent summaries), including zero-range processes of the type introduced in [13, 14], inclusion processes with a rescaled system parameter [15, 16] and explosive condensation models [17, 18]. While the stationary measures have been understood in great detail on a rigorous level [12, 19, 20, 21, 22], the dynamics of these processes continue to pose interesting mathematical questions. First recent results for zero-range and inclusion processes have been obtained on metastability in the stationary dynamics of the condensate location [23, 24, 25], approach to stationarity on fixed lattices under diverging particle density [15, 26], and a hydrodynamic limit for density profiles below the critical value [27].

Our result provides a contribution towards a rigorous understanding of the approach to stationarity in the thermodynamic limit of diverging system size and particle number. This exhibits an interesting coarsening regime characterized by a power-law time evolution of typical observables, which has been identified in previous heuristic results [1, 20, 18, 28] also on finite dimensional regular lattices. Condensation implies that stationary measures for the limiting birth death dynamics only exist up to a first moment ρ_c , above which $f_k(t)$ phase separates over time into two parts describing the mass distribution in the condensate and the background of the underlying particle system. Explicit travelling wave scaling solutions for the condensed part of the distribution have been found in [1, 29, 28] for zero-range processes and in [5] for a specific inclusion process, and will be discussed in detail.

The paper is organized as follows. In Section 2 we introduce notation and state our main result with the proof given in Section 3. In Section 4 we discuss basic properties of the limit dynamics and its solutions, as well as limitations and possible extensions of our result. We present particular examples of condensing systems in Section 5 and provide a concluding discussion in Section 6.

2 Notation and Main result

We consider a stochastic particle system $(\eta(t) : t \geq 0)$ of misanthrope type [8] on finite lattices Λ of size $|\Lambda| = L$. Configurations are denoted by $\eta = (\eta_x : x \in \Lambda)$ where $\eta_x \in \mathbb{N}_0$ is the number

of particles on site x , and the state space is denoted by $\Omega = \mathbb{N}_0^\Lambda$. The dynamics of the process is defined by the infinitesimal generator

$$(\mathcal{L}h)(\boldsymbol{\eta}) = \sum_{x,y \in \Lambda} q(x,y)c(\eta_x, \eta_y)(h(\boldsymbol{\eta}^{x \rightarrow y}) - h(\boldsymbol{\eta})). \quad (1)$$

Here the usual notation $\boldsymbol{\eta}^{x \rightarrow y}$ indicates a configuration where one particle has moved from site x to y , i.e. $\eta_z^{x \rightarrow y} = \eta_z - \delta_{z,x} + \delta_{z,y}$, and δ is the Kronecker delta. To ensure that the process is non-degenerate, the jump rates satisfy

$$\begin{cases} c(0, l) = 0 & \text{for all } l \geq 0 \\ c(k, l) > 0 & \text{for all } k > 0 \text{ and } l \geq 0. \end{cases} \quad (2)$$

Since we focus on finite lattices only, the generator (1) is defined for all bounded, continuous test functions $h \in C^b(\Omega)$. For a general discussion and the construction of the dynamics on infinite lattices see [30, 31]. We focus on complete graph dynamics, i.e. $q(x, y) = 1/(L-1)$ for all $x \neq y$, and denote by \mathbb{P}^L and \mathbb{E}^L the law and expectation on the path space $D_{[0,\infty)}(\Omega)$ of the process.

As usual, we use the Borel σ -algebra for the discrete product topology on Ω , and the smallest σ -algebra on $D_{[0,\infty)}(\Omega)$ such that $\omega \mapsto \eta_t(\omega)$ is measurable for all $t \geq 0$. We will study the processes $t \mapsto F_k(\boldsymbol{\eta}(t))$ defined by the test functions

$$F_k(\boldsymbol{\eta}) := \frac{1}{L} \sum_{x \in \Lambda} \delta_{\eta_x, k} \in [0, 1], \quad (3)$$

counting the fraction of lattice sites for each occupation number $k \geq 0$. Expectations are denoted by

$$f_k^L(t) := \mathbb{E}^L[F_k(\boldsymbol{\eta}(t))] = \frac{1}{L} \sum_{x \in \Lambda} \mathbb{P}^L[\eta_x(t) = k] \in [0, 1], \quad (4)$$

and we write $f^L(t) = (f_k^L(t) : k \in \mathbb{N}_0)$. Note that $f^L(t)$ are probability distributions on \mathbb{N}_0 for all $t \geq 0$. The time evolution is then given by

$$\frac{d}{dt} f_k^L(t) = \frac{d}{dt} \mathbb{E}[F_k(\boldsymbol{\eta}(t))] = \mathbb{E}[(\mathcal{L}F_k)(\boldsymbol{\eta}(t))], \quad (5)$$

and as usual this equation is not closed for finite system sizes L , since the right-handed side is not a function of $f^L(t)$. Our aim is to derive a closed equation in the limit $L \rightarrow \infty$.

In the following, we consider a sequence (in L) of initial conditions $(\eta_x(0) : x \in \Lambda)$ of the process such that

$$\{\eta_x(0) : x \in \Lambda\} \quad \text{are i.i.d. with distribution } f^L(0) \text{ for all } L, \quad (6)$$

and such that there exists a probability distribution $f(0) = (f_k(0) : k \in \mathbb{N}_0)$ with

$$f_k^L(0) \rightarrow f_k(0) \text{ for all } k \geq 0 \text{ as } L \rightarrow \infty. \quad (7)$$

The second condition excludes cases where the sequence $f^L(0)$ is not tight or does not have a unique limit. The simplest choice with the required properties is, of course, a product measure with marginals $f^L(0) = f(0)$ for all L . By symmetry of the dynamics on the complete graph, $(\eta_x(t) : x \in \Lambda)$ is therefore permutation invariant for all $t \geq 0$ and we also have

$$f_k^L(t) = \mathbb{P}^L[\eta_x(t) = k] \quad \text{for all } x \in \Lambda. \quad (8)$$

We further assume that the jump rates are uniformly bounded with

$$\bar{C} := \sup_{k,l} c(k, l) < \infty. \quad (9)$$

Our main theorem can be formulated as a convergence result for the single site dynamics with state space \mathbb{N}_0 ,

$$(\eta_x(t) : t \geq 0) \quad \text{for fixed } x \in \Lambda \text{ (with } \Lambda \text{ big enough)}. \quad (10)$$

Theorem 1. *Consider a process with generator (1) on the complete graph with uniformly bounded rates (9) and initial conditions satisfying (6) and (7). Then, the single site process (10) converges weakly on path space $D_{[0,\infty)}(\mathbb{N}_0)$ to a birth death chain with distribution $f(t) = (f_k(t) : k \in \mathbb{N}_0)$ characterized by the **mean-field master equation***

$$\begin{aligned} \frac{df_k(t)}{dt} = & \sum_{l \geq 0} c(k+1, l) f_l(t) f_{k+1}(t) + \sum_{l \geq 0} c(l, k-1) f_l(t) f_{k-1}(t) \\ & - \left(\sum_{l \geq 0} c(k, l) f_l(t) + \sum_{l \geq 0} c(l, k) f_l(t) \right) f_k(t) \quad \text{for all } k \geq 0, \end{aligned} \quad (11)$$

with initial condition $f(0)$ given by (7). Here we use the convention $f_{-1}(t) \equiv 0$ for all $t \geq 0$ and recall that $c(0, l) = 0$ for all $l \geq 0$. (11) has a unique solution $(f(t) : t \geq 0)$, and in particular $f^L(t) \rightarrow f(t)$ weakly as $L \rightarrow \infty$ for all $t \geq 0$.

We see that $\frac{d}{dt} \sum_{k \geq 0} f_k(t) = 0$, and with (7) the limit is indeed as the master equation of a birth death chain with state space \mathbb{N}_0 , birth rate $\sum_{l \geq 0} c(l, k) f_l(t)$ and death rate $\sum_{l \geq 0} c(k, l) f_l(t)$. Note that the chain and its master equation are non-linear since the birth and death rates depend on the distribution $f(t)$. Further details are provided in Section 6.

3 Proof of the main result

The proof follows a standard approach. We first establish existence of limits via a tightness argument, then characterize all limit points as solutions of (11) using a coupling to a branching process based on the graphical construction, and finally show that (11) has a unique solution for a given initial condition.

3.1 Existence

Proposition 1. *Consider the process with generator (1) and conditions as in Theorem 1. Then, the law of the single site process $\eta_x(t)$ (10) is tight as $L \rightarrow \infty$. This implies existence of weak limit points $(f(t) : t \in [0, T])$ of the sequence $(f^L(t) : t \in [0, T])$ as defined in (4) for all fixed $T > 0$.*

Proof. For each L large enough, consider the single site process $\eta_x(t)$ for a fixed $x \in \Lambda$ with law \mathbb{Q}^L on the path space $D_{[0, \infty)}(\mathbb{N}_0)$. We will show tightness of the sequence \mathbb{Q}^L as $L \rightarrow \infty$, which implies existence of limit points \mathbb{Q} . Since $f_k^L(t) = \mathbb{Q}^L[\eta_x(t) = k]$, this also provides existence of limit points $t \mapsto f(t) = \mathbb{Q}[\eta_x(t) = \cdot]$.

Interpreting $\eta_x : \boldsymbol{\eta} \mapsto \eta_x$ as a mapping, $\mathbb{Q}^L = \mathbb{P}^L \circ \eta_x^{-1}$ is given as the image measure of \mathbb{P}^L under η_x . By a version of Aldous' criterion to establish tightness for \mathbb{Q}^L (cf. Theorem 16.10 in [32]), it suffices to show that for any $\epsilon > 0$

$$\lim_{\delta \rightarrow 0^+} \limsup_{L \rightarrow \infty} \sup_{s < \delta} \sup_{\zeta \in \Omega} \mathbb{P}_{\zeta}^L(|\eta_x(s) - \zeta_x| > \epsilon) = 0. \quad (12)$$

Here $\zeta \in \Omega$ denotes the initial condition of the original process and \mathbb{P}_{ζ}^L the corresponding path measure. For fixed ζ and x from above, consider the test function $f(\boldsymbol{\eta}) = |\eta_x - \zeta_x|$ to get

$$\begin{aligned} \mathcal{L}f(\boldsymbol{\eta}) &= \frac{1}{L-1} \sum_{y \neq x} \left[c(\eta_y, \eta_x) (|\eta_x - \zeta_x + 1| - |\eta_x - \zeta_x|) \right. \\ &\quad \left. + c(\eta_x, \eta_y) (|\eta_x - \zeta_x - 1| - |\eta_x - \zeta_x|) \right] \\ &= \frac{1}{L-1} \left(\sum_{y \neq x} c(\eta_y, \eta_x) - \sum_{y \neq x} c(\eta_x, \eta_y) \right) (\mathbb{I}_{\eta_x \geq \zeta_x}(\boldsymbol{\eta}) - \mathbb{I}_{\eta_x < \zeta_x}(\boldsymbol{\eta})), \end{aligned} \quad (13)$$

with standard notation for indicator functions \mathbb{I} . By Itô's formula and with $f(\boldsymbol{\eta}(0)) = 0$, we have for any $s > 0$

$$\begin{aligned} |\eta_x(s) - \zeta_x| &= \int_0^s \frac{1}{L-1} \left(\sum_{y \neq x} c(\eta_y(u), \eta_x(u)) - \sum_{y \neq x} c(\eta_x(u), \eta_y(u)) \right) \\ &\quad (\mathbb{I}_{\eta_x \geq \zeta_x}(\boldsymbol{\eta}(u)) - \mathbb{I}_{\eta_x < \zeta_x}(\boldsymbol{\eta}(u))) du + M(s), \end{aligned} \quad (14)$$

where $(M(s) : s > 0)$ is a martingale. It has quadratic variation

$$[M](s) = \int_0^s [\mathcal{L}f^2 - 2f\mathcal{L}f](\boldsymbol{\eta}(u)) du,$$

and the integrand is easily computed to be

$$[\mathcal{L}f^2 - 2f\mathcal{L}f](\boldsymbol{\eta}) = \frac{1}{L-1} \left(\sum_{y \neq x} c(\eta_y, \eta_x) + \sum_{y \neq x} c(\eta_x, \eta_y) \right).$$

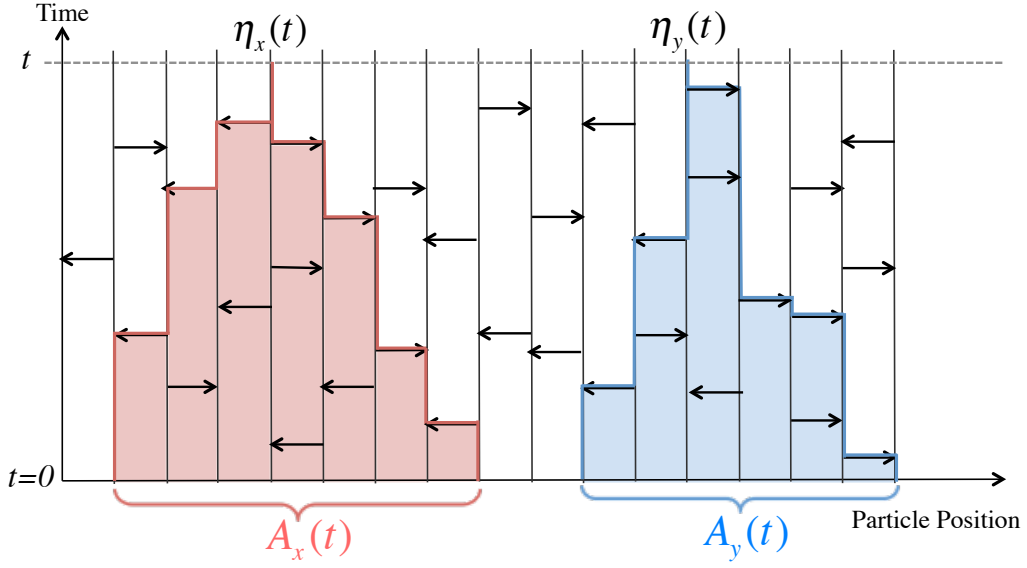


Figure 1: Illustration of the graphical construction of the process (1) for one-dimensional nearest neighbour dynamics. It is based on independent Poisson processes $PP(\bar{C}/2)$ with jump events shown as \rightarrow and \leftarrow . The sets $A_x(t)$ and $A_y(t)$ as given in Lemma 1, possibly influencing $\eta_x(t)$ and $\eta_y(t)$, respectively, are shown in red and blue.

Since the rates are bounded (9), we have for the first term in (14)

$$\left| \int_0^s \frac{1}{L-1} \left(\sum_{y \neq x} c(\eta_y(u), \eta_x(u)) - \sum_{y \neq x} c(\eta_x(u), \eta_y(u)) \right) du \right| \leq 2\bar{C}s \rightarrow 0 \quad (15)$$

as $s \rightarrow 0$, which holds \mathbb{P}_ζ^L -a.s. uniformly in $\zeta \in \Omega$ and in L . The same argument applies to the quadratic variation part, where for $s \rightarrow 0$ we get

$$[M](s) = \int_0^s \frac{1}{L-1} \left(\sum_{y \neq x} c(\eta_y(u), \eta_x(u)) + \sum_{y \neq x} c(\eta_x(u), \eta_y(u)) \right) du \leq 2\bar{C}s \rightarrow 0. \quad (16)$$

Almost sure convergence in (15) and (16) uniformly in $\zeta \in \Omega$ and in L implies (12). Therefore, limit points $(f(t) : t \in [0, T])$ exist for all compact time interval following from the usual topology of weak convergence on path space $D_{[0, \infty)}(\mathbb{N}_0)$. \square

3.2 Characterization of limit points

Proposition 2. *Consider the process with generator (1) and conditions as in Theorem 1. Every limit point $(f(t) : t \in [0, T])$ of Proposition 1 satisfies the mean-field rate equation (11).*

We first collect some auxiliary results before giving the proof. Recall the standard graphical construction of interacting particle systems [33], which consists of a family of independent Poisson point processes $PP_{xy}(\frac{\bar{C}}{L-1})$ for each pair $x \neq y \in \Lambda$. For a given η , at the jump time of the point process a particle jumps from x to y with probability $c(\eta_x, \eta_y)/\bar{C}$. This is illustrated in Figure 1 for one-dimensional nearest neighbour dynamics.

We say $(y, 0)$ is connected to (x, t) , writing $(y, 0) \rightarrow (x, t)$, if there exists a (forward in time) directed path along jump events in $\cup_{z, z' \in \Lambda} PP_{zz'} \left(\frac{\bar{C}}{L-1} \right)$ from $(y, 0)$ to (x, t) . Equivalently, consider running a contact process without recovery backward in time using all jump events of $\cup_{z, z' \in \Lambda} PP_{zz'} \left(\frac{\bar{C}}{L-1} \right)$ in the time interval $[0, t]$, starting with a single infection at site x . Then $(y, 0) \rightarrow (x, t)$ if y is infected at time 0. We write

$$A_x(t) = \{y \in \Lambda : (y, 0) \rightarrow (x, t)\}, \quad (17)$$

for all sites whose configuration at time 0 possibly influences $\eta_x(t)$. See Figure 1 for an illustration in one dimension. Using the graphical construction, the (backward in time) contact process

$$\tau \mapsto A_x(t, \tau) := \{y \in \Lambda : (y, t - \tau) \rightarrow (x, t)\}, \quad \tau \in [0, t] \quad (18)$$

can be coupled to a pure birth process $(N(t) : t \geq 0)$ with state space \mathbb{N} and generator

$$\mathcal{L}f(n) = n\bar{C}(f(n+1) - f(n)) \quad \text{for } f : \mathbb{N} \rightarrow \mathbb{R}, \quad (19)$$

such that

$$|A_x(t, \tau)| \leq N(\tau) \text{ for all } \tau \in [0, t].$$

Note that inequality holds since already infected sites cannot be infected again and the (forward) process $t \mapsto A_x(t)$ is increasing and $A_x(0) = \{x\}$.

Lemma 1. *Consider $A_x(t)$ and $A_y(t)$ as defined above with $y \neq x$. Then, for each fixed $t \geq 0$,*

$$\mathbb{P}^L[A_x(t) \cap A_y(t) = \emptyset] \rightarrow 1 \quad \text{as } L \rightarrow \infty. \quad (20)$$

Proof. It is immediate from the graphical construction and symmetry of the dynamics that conditioned on their sizes $n_x, n_y \geq 1$, $A_x(t)|_{|A_x(t)|=n_x}$ and $A_y(t)|_{|A_y(t)|=n_y}$ are uniform subsets of Λ . It is further immediate, that both processes evolve independently until the first intersection time

$$T = \inf \{t > 0 : A_x(t) \cap A_y(t) \neq \emptyset\} \in (0, \infty),$$

and that

$$\{T > t\} = \{A_x(t) \cap A_y(t) = \emptyset\}.$$

Thus, we have for all fixed $t > 0$ and $n_x, n_y \geq 1$

$$\begin{aligned} & \mathbb{P}^L[A_x(t) \cap A_y(t) = \emptyset \mid |A_x(t)| = n_x, |A_y(t)| = n_y] \\ &= \frac{L - n_x - 1}{L - 1} \cdot \frac{L - n_y - 1}{L - 2} \cdots \frac{L - n_x - n_y}{L - n_y} \rightarrow 1, \quad \text{as } L \rightarrow \infty. \end{aligned} \quad (21)$$

Also, for $t < T$, we can use the coupling to compare two independent copies $N_x(t)$ and $N_y(t)$ of

a pure birth process (19) with law denoted by \mathbb{P} , to get

$$\frac{\mathbb{P}^L[|A_x(t)| = n_x, |A_y(t)| = n_y, T > t]}{\mathbb{P}[N_x(t) = n_x] \mathbb{P}[N_y(t) = n_y]} \rightarrow 1 \quad \text{as } L \rightarrow \infty, \quad (22)$$

since the probability of attempted double infection of a given site x in the contact process (18) vanishes as $L \rightarrow \infty$. Therefore, since $N_x(t)$ is \mathbb{P} -a.s. finite¹, we get

$$\begin{aligned} \mathbb{P}^L[A_x(t) \cap A_y(t) = \emptyset] &= \sum_{n_x, n_y=0}^{\infty} H_L(n_x, n_y) \cdot \mathbb{P}[N_x(t) = n_x] \mathbb{P}[N_y(t) = n_y] \\ &\rightarrow \sum_{n_x, n_y=0}^{\infty} \mathbb{P}[N_x(t) = n_x] \mathbb{P}[N_y(t) = n_y] = 1, \end{aligned}$$

as $L \rightarrow \infty$ by dominated convergence. Here we used that the integrand

$$\begin{aligned} H_L(n_x, n_y) &= \mathbb{P}^L[A_x(t) \cap A_y(t) = \emptyset \mid |A_x(t)| = n_x, |A_y(t)| = n_y] \mathbb{I}_{n_x \leq L} \mathbb{I}_{n_y \leq L} \\ &\cdot \frac{\mathbb{P}^L[|A_x(t)| = n_x, |A_y(t)| = n_y, T > t]}{\mathbb{P}[N_x(t) = n_x] \mathbb{P}[N_y(t) = n_y]} \rightarrow 1, \end{aligned}$$

as $L \rightarrow \infty$ for all fixed $n_x, n_y \in \mathbb{N}$, following from (21) and (22). \square

Proof of Proposition 2. Applying the generator (1) with $q(x, y) = \frac{1}{L-1}$ to the test function F_k , we get

$$\begin{aligned} (\mathcal{L}F_k)(\boldsymbol{\eta}) &= \sum_{x, y \in \Lambda} \frac{1}{L-1} c(\eta_x, \eta_y) [F_k(\boldsymbol{\eta}^{x \rightarrow y}) - F_k(\boldsymbol{\eta})] \\ &= -\frac{1}{L-1} \sum_{x, y \in \Lambda} c(k, \eta_y) \frac{\delta_{k, \eta_x}}{L} + \frac{1}{L-1} \sum_{x \in \Lambda} c(k, \eta_x) \frac{\delta_{k, \eta_x}}{L} \\ &\quad - \frac{1}{L-1} \sum_{x, y \in \Lambda} c(\eta_x, k) \frac{\delta_{k, \eta_y}}{L} + \frac{1}{L-1} \sum_{x \in \Lambda} c(\eta_x, k) \frac{\delta_{k, \eta_x}}{L} \\ &\quad + \frac{1}{L-1} \sum_{x, y \in \Lambda} c(\eta_x, k-1) \frac{\delta_{k-1, \eta_y}}{L} - \frac{1}{L-1} \sum_{x \in \Lambda} c(\eta_x, k-1) \frac{\delta_{k-1, \eta_x}}{L} \\ &\quad + \frac{1}{L-1} \sum_{x, y \in \Lambda} c(k+1, \eta_y) \frac{\delta_{k+1, \eta_x}}{L} - \frac{1}{L-1} \sum_{x \in \Lambda} c(k+1, \eta_x) \frac{\delta_{k+1, \eta_x}}{L} \\ &= -\frac{1}{L-1} \sum_{y \in \Lambda} c(k, \eta_y) F_k(\boldsymbol{\eta}) - \frac{1}{L-1} \sum_{x \in \Lambda} c(\eta_x, k) F_k(\boldsymbol{\eta}) \\ &\quad + \frac{1}{L-1} \sum_{x \in \Lambda} c(\eta_x, k-1) F_{k-1}(\boldsymbol{\eta}) + \frac{1}{L-1} \sum_{y \in \Lambda} c(k+1, \eta_y) F_{k+1}(\boldsymbol{\eta}) \\ &\quad + \frac{1}{L-1} \left(-(B_{L,k} + B'_{L,k}) + B'_{L,k-1} + B_{L,k+1} \right). \end{aligned} \quad (23)$$

¹A standard computation with generating functions in fact reveals that $N(t)$ has a geometric distribution, where $\mathbb{P}[N(t) = n] = e^{-\bar{C}t} (1 - e^{-\bar{C}t})^{n-1}$.

Here $B_{L,k} := \sum_x c(\eta_x, k) F_k(\boldsymbol{\eta})$ and $B'_{L,k} := \sum_x c(k, \eta_x) F_k(\boldsymbol{\eta})$ are corrections resulting from diagonal terms in the sum over $x, y \in \Lambda$, and are uniformly bounded in k and L .

In the following, we will show that $f_k(t)$ fulfills (11). From (23) and (4)-(5), we have

$$\begin{aligned} \frac{d}{dt} f_k^L(t) &= -\mathbb{E}^L \left[\frac{1}{L} \sum_{y \in \Lambda} c(k, \eta_y) F_k(\boldsymbol{\eta}) \right] - \mathbb{E}^L \left[\frac{1}{L} \sum_{x \in \Lambda} c(\eta_x, k) F_k(\boldsymbol{\eta}) \right] \\ &\quad + \mathbb{E}^L \left[\frac{1}{L} \sum_{x \in \Lambda} c(\eta_x, k-1) F_{k-1}(\boldsymbol{\eta}) \right] + \mathbb{E}^L \left[\frac{1}{L} \sum_{y \in \Lambda} c(k+1, \eta_y) F_{k+1}(\boldsymbol{\eta}) \right] \\ &\quad + O(1/L), \end{aligned} \tag{24}$$

where we replaced pre-factors $1/(L-1)$ by $1/L$ at the expense of a further correction of order $O(1/L)$. To conclude, we will establish that expectations of product terms in (24) factorize. For the second term, we have

$$\begin{aligned} \frac{1}{L} \sum_{x \in \Lambda} c(\eta_x, k) F_k(\boldsymbol{\eta}) &= \sum_{l \geq 0} c(l, k) \frac{1}{L} \sum_{x \in \Lambda} \delta_{\eta_x, l} \frac{1}{L} \sum_{y \in \Lambda} \delta_{\eta_y, k} \\ &= \sum_{l \geq 0} c(l, k) \frac{1}{L^2} \sum_{x, y \in \Lambda} \delta_{\eta_x, l} \delta_{\eta_y, k} \\ &= \sum_{l \geq 0} c(l, k) \frac{1}{L^2} \sum_{x, y \neq x} \delta_{\eta_x, l} \delta_{\eta_y, k} + c(k, k) \frac{1}{L^2} \sum_{x \in \Lambda} \delta_{\eta_x, k}. \end{aligned}$$

Since the rates are bounded (9) and $F_k(\boldsymbol{\eta}) \leq 1$, we have

$$\begin{aligned} \mathbb{E}^L \left[\frac{1}{L} \sum_{x \in \Lambda} c(\eta_x, k) F_k(\boldsymbol{\eta}) \right] &= \sum_{l \geq 0} c(l, k) \frac{1}{L^2} \sum_{x \neq y} \mathbb{P}^L[\eta_x(t) = l, \eta_y(t) = k] + O(1/L) \\ &= \sum_{l \geq 0} c(l, k) \mathbb{P}^L[\eta_x(t) = l, \eta_y(t) = k] + O(1/L), \end{aligned} \tag{25}$$

where we can fix particular sites $x \neq y$ in the last line by symmetry of the process.

Now, in order to use Lemma 1 we write

$$\begin{aligned} \mathbb{P}^L[\eta_x(t) = k, \eta_y(t) = l] &= \mathbb{P}^L[\eta_x(t) = k, \eta_y(t) = l, A_x(t) \cap A_y(t) = \emptyset] \\ &\quad + \mathbb{P}^L[\eta_x(t) = k, \eta_y(t) = l, A_x(t) \cap A_y(t) \neq \emptyset], \end{aligned}$$

and as $L \rightarrow \infty$ we have for the second term

$$\mathbb{P}^L[\eta_x(t) = k, \eta_y(t) = l, A_x(t) \cap A_y(t) \neq \emptyset] \leq \mathbb{P}^L[A_x(t) \cap A_y(t) \neq \emptyset] \rightarrow 0.$$

For the first term we write

$$\begin{aligned} \mathbb{P}^L[\eta_x(t) = k, \eta_y(t) = l, A_x(t) \cap A_y(t) = \emptyset] \\ = \mathbb{P}^L[\eta_x(t) = k, \eta_y(t) = l \mid A_x(t) \cap A_y(t) = \emptyset] \mathbb{P}^L[A_x(t) \cap A_y(t) = \emptyset], \end{aligned}$$

where $\mathbb{P}^L[A_x(t) \cap A_y(t) = \emptyset] \rightarrow 1$ as $L \rightarrow \infty$ by Lemma 1. Conditional on $\{A_x(t) \cap A_y(t) = \emptyset\}$, the events $\{\eta_x(t) = k\}$ and $\{\eta_y(t) = l\}$ are independent by construction and independence of initial conditions (6), and therefore

$$\begin{aligned} & \mathbb{P}^L[\eta_x(t) = k, \eta_y(t) = l \mid A_x(t) \cap A_y(t) = \emptyset] \\ &= \mathbb{P}^L[\eta_x(t) = k \mid A_x(t) \cap A_y(t) = \emptyset] \mathbb{P}^L[\eta_y(t) = l \mid A_x(t) \cap A_y(t) = \emptyset] \\ &\rightarrow f_k(t)f_l(t), \end{aligned} \tag{26}$$

as $L \rightarrow \infty$. Convergence to the limit points f_k and f_l uses again that the conditional event has limiting probability 1 with Lemma 1. With bounded convergence in (25) this implies factorization of

$$\mathbb{E}^L \left[\sum_{x \in \Lambda} c(\eta_x, k) F_k(\eta) \right] \rightarrow \sum_{l \geq 0} c(l, k) f_l(t) f_k(t), \text{ as } L \rightarrow \infty,$$

which follows analogously for the other terms in (24). This completes the proof. \square

3.3 Uniqueness

We consider solutions of (11) $f(t) = (f_k(t) : k \in \mathbb{N}_0)$ which are limit points of the sequence $f^L(t)$. Since $\sum_{k \geq 0} f_k^L(t) = 1$ and $f_k^L \in [0, 1]$ for all $t \geq 0$ and L , $f_k(t) \in [0, 1]$ for all $t \geq 0$ and $k \geq 0$. Furthermore, Fatou's Lemma implies

$$\|f(t)\|_1 = \sum_{k \geq 0} f_k(t) \leq 1 \quad \text{and therefore } f \in \ell^1.$$

This also implies $f(t) \in \ell^2$ with $\|f(t)\|_2^2 = \sum_{k \geq 0} f_k^2(t) < \infty$.

Let Q be non-linear operator with $(Qf)_l = \sum_{k \geq 0} c(k, l) f_k$ for all $f \in \ell^1 \cap \ell^2$. Since $0 \leq c(k, l) \leq \bar{C} < \infty$, we have

$$\|Qf\|_1 \leq \bar{C}\|f\|_1 \text{ and } \|Qf\|_2 \leq \bar{C}\|f\|_2.$$

Proposition 3. *Let $t \mapsto f(t)$ be a solution to (11) with $f_k(t) \in [0, 1]$ and $\sum_k f_k(t) \leq 1$ for all $k \geq 0, t \geq 0$. Then, $t \mapsto f(t)$ is unique.*

Proof. Suppose f and \hat{f} are two solutions of (11) with above properties and $f(0) = \hat{f}(0)$. With

the convention $f_{-1} = \hat{f}_{-1} \equiv 0$, we have

$$\begin{aligned} \frac{d}{dt} \|f - \hat{f}\|_2^2 &= 2 \sum_{k \geq 0} (f_k - \hat{f}_k) \frac{d}{dt} \sum_{k \geq 0} (f_k - \hat{f}_k) \\ &= 2 \sum_{k \geq 0} (f_k - \hat{f}_k) \left[\sum_{l \geq 0} c(l, k-1) (f_{k-1} f_l - \hat{f}_{k-1} \hat{f}_l) \right. \\ &\quad \left. + \sum_{l \geq 0} c(k+1, l) (f_{k+1} f_l - \hat{f}_{k+1} \hat{f}_l) \right. \\ &\quad \left. - \sum_{l \geq 0} (c(k, l) + c(l, k)) (f_k f_l - \hat{f}_k \hat{f}_l) \right], \end{aligned}$$

omitting the time argument of f to simplify notation. In the following we use

$$f_k f_l - \hat{f}_k \hat{f}_l = f_k (f_l - \hat{f}_l) + \hat{f}_l (f_k - \hat{f}_k)$$

together with boundedness of shift operator $(Sf)_k = f_{k+1}$, i.e. $\|Sf\|_2^2 \leq \|f\|_2^2$, and the Cauchy-Schwarz inequality

$$\langle g, h \rangle := \sum_{k \geq 0} g_k h_k \leq \|g\|_2 \|h\|_2 \quad \text{for all } g, h \in \ell^2.$$

We get

$$\begin{aligned} \frac{d}{dt} \|f - \hat{f}\|_2^2 &\leq 2 \left[\sum_{k, l \geq 1} (f_k - \hat{f}_k) c(l, k-1) (f_{k-1} (f_l - \hat{f}_l) + (f_{k-1} - \hat{f}_{k-1}) \hat{f}_l) \right. \\ &\quad \left. + \sum_{k, l \geq 0} (f_k - \hat{f}_k) c(k+1, l) (f_{k+1} (f_l - \hat{f}_l) + (f_{k+1} - \hat{f}_{k+1}) \hat{f}_l) \right. \\ &\quad \left. + \sum_{k, l \geq 0} (f_k - \hat{f}_k) (c(k, l) + c(l, k)) (f_k (f_l - \hat{f}_l) + (f_k - \hat{f}_k) \hat{f}_l) \right] \\ &\leq 2 \left[\langle S(f - \hat{f}), Q(f - \hat{f}) \rangle + \bar{C} \|f - \hat{f}\|_2^2 \right. \\ &\quad \left. + \langle f - \hat{f}, SQ(f - \hat{f}) \rangle + \bar{C} \langle f - \hat{f}, S(f - \hat{f}) \rangle \right. \\ &\quad \left. + 2 \langle f - \hat{f}, Q(f - \hat{f}) \rangle + 2\bar{C} \|f - \hat{f}\|_2^2 \right] \\ &\leq 16 \bar{C} \|f - \hat{f}\|_2^2, \end{aligned}$$

having also used $f_k, \hat{f}_k \leq 1$ for all $k \geq 0$ to get the second inequality. Since we assume the initial condition $f(0) - \hat{f}(0) = 0$, by Gronwall's inequality we get

$$\|f(t) - \hat{f}(t)\|_2^2 \leq \|f(0) - \hat{f}(0)\|_2^2 \exp(16 \bar{C} t) = 0 \quad \text{for all } t \geq 0.$$

Hence, $f(t) = \hat{f}(t)$ for all $t \geq 0$ and the solution $t \mapsto f(t)$ is unique. \square

4 Properties of solutions

4.1 Conserved quantities

Since $f_k(t)$ are limits of $f_k^L(t) \in [0, 1]$, we have $f_k(t) \in [0, 1]$ for all $k \geq 0, t \geq 0$. We denote the i^{th} moment of $f(t)$ by

$$m_i(t) = \sum_{k \geq 0} k^i f_k(t).$$

The limiting mean-field equation (11) is the master equation of the non-linear birth death chain $(X_t : t \geq 0)$ on \mathbb{N}_0 with generator

$$\mathcal{L}_{BD}h(k) = \sum_{l \geq 0} c(k, l) f_l(t) (h(k+1) - h(k)) + \sum_{l \geq 0} c(l, k) f_l(t) (h(k-1) - h(k)), \quad (27)$$

where $c(0, l) = 0$ for all $l \geq 0$. This is the limit dynamics of the single site process (10), and the time dependent birth rates $\beta_k(t)$ and death rates $\mu_k(t)$ are given by

$$\beta_k(t) = \sum_{l \geq 0} c(l, k) f_l(t) \quad \text{and} \quad \mu_k(t) = \sum_{l \geq 0} c(k, l) f_l(t). \quad (28)$$

Note that this immediately implies that $f_k = \delta_{0,k}$ is stationary, but in general 0 is not an absorbing state as long as $f_k(0) > 0$ for some $k > 0$, as discussed in detail later. The adjoint operator \mathcal{L}_{BD}^\dagger then characterizes the right-hand side of the master equation (11) which can be written as

$$\frac{d}{dt} f(t) = \mathcal{L}_{BD}^\dagger f(t).$$

$f(t)$ is indeed a probability distribution on \mathbb{N}_0 for all $t \geq 0$ since we have

$$\mathcal{L}_{BD}1 = 0 \quad \text{and therefore} \quad m_0(t) = m_0(0) = 1.$$

Also, as usual $\mathcal{L}_{BD}k = \beta_k(t) - \delta_k(t)$, which leads to

$$\frac{d}{dt} m_1(t) = \sum_{k \geq 0} f_k(t) \mathcal{L}_{BD}k = \sum_{k \geq 0} \sum_{l \geq 0} f_k(t) f_l(t) (c(l, k) - c(k, l)) = 0.$$

This implies that the expectation is conserved for the chain $(X_t : t \geq 0)$, i.e.

$$m_1(t) = m_1(0) =: \rho > 0,$$

which corresponds to the particle density ρ in the original particle system. Note, however, that $(X_t : t \geq 0)$ is not a martingale since $\mathcal{L}_{BD}k \neq 0$, and the conservation of m_1 results from the non-linearity of the process. By assumption (2) on the rates c the chain is further irreducible unless $f(0)$ is degenerate, but we will see below that the additional conserved quantity leads to non-uniqueness for the stationary distribution.

4.2 Stationary distributions

A misanthrope-type process with generator (1) on the complete graph has a family of stationary product measures ν_ϕ , provided that

$$\frac{c(k, l)}{c(l+1, k-1)} = \frac{c(k, 0)c(1, l)}{c(l+1, 0)c(1, k-1)} \quad \text{for all } k \geq 1, l \geq 0. \quad (29)$$

This is well known (see e.g. [3, 8, 34]) also for more general translation invariant dynamics under additional conditions on c . The marginals are given explicitly by

$$\nu_\phi[\eta_x = n] = \frac{1}{z(\phi)} w(n) \phi^n \quad \text{with} \quad w(n) = \prod_{k=1}^n \frac{c(1, k-1)}{c(k, 0)}, \quad (30)$$

which are normalized by the partition function

$$z(\phi) := \sum_{n=0}^{\infty} w(n) \phi^n.$$

The parameter $\phi \geq 0$ is the fugacity controlling the average particle density

$$R(\phi) := \sum_{n=0}^{\infty} n \nu_\phi[\eta_x = n] = \phi \partial_\phi \log(z(\phi)), \quad (31)$$

which is a monotone increasing function of ϕ with $R(0) = 0$. These distributions exist for all $\phi \in D_\phi = \{\phi \geq 0 : z(\phi) < \infty\}$. The domain is of the form $D_\phi = [0, \phi_c]$ or $[0, \phi_c)$, where

$$\phi_c = (\limsup_{n \rightarrow \infty} w(n)^{1/n})^{-1}$$

is the radius of convergence of $z(\phi)$, and we denote by

$$\rho_c = R(\phi_c) \in [0, \infty] \quad (32)$$

the maximal density for the family of product measures. If $D_\phi = [0, \phi_c)$ then $\rho_c = \infty$, and if $\rho_c < \infty$ the model exhibits a condensation transition (see e.g. [12]), which we discuss in more detail in Section 5.

Then, for each $\phi \in D_\phi$, the single site marginal $f_k^\phi := \nu_\phi[\eta_x = k]$ is a stationary solution of (11). From (30), we have the relation

$$\frac{c(k, 0)}{c(1, k-1)} f_k^\phi = \phi f_{k-1}^\phi \quad \text{for all } k \geq 1 \text{ and } \phi \in D_\phi. \quad (33)$$

With the usual convention $f_{-1}^\phi = 0$ and $c(0, l) = 0$ for all $l \geq 0$ this leads to

$$\begin{aligned}
\mathcal{L}_{BD}^\dagger f_k^\phi &= \sum_{l \geq 0} c(k+1, l) f_l^\phi f_{k+1}^\phi + \sum_{l \geq 1} c(l, k-1) f_l^\phi f_{k-1}^\phi \\
&\quad - \left(\sum_{l \geq 1} c(l, k) f_l^\phi + \sum_{l \geq 0} c(k, l) f_l^\phi \right) g f_k^\phi \\
&= \sum_{l \geq 1} c(k+1, l-1) f_{l-1}^\phi \frac{c(1, k)}{c(k+1, 0)} \phi f_k^\phi + \sum_{l \geq 0} c(l+1, k-1) f_{l+1}^\phi \frac{c(k, 0)}{c(1, k-1)} \frac{1}{\phi} f_k^\phi \\
&\quad - \left(\sum_{l \geq 1} c(l, k) f_l^\phi + \sum_{l \geq 0} c(k, l) f_l^\phi \right) f_k^\phi \\
&= \sum_{l \geq 1} c(l, k) \frac{c(1, l-1)}{c(l, 0)} \phi f_{l-1}^\phi f_k^\phi + \sum_{l \geq 0} c(k, l) \frac{c(l+1, 0)}{c(1, l)} \frac{1}{\phi} f_{l+1}^\phi f_k^\phi \\
&\quad - \left(\sum_{l \geq 1} c(l, k) f_l^\phi + \sum_{l \geq 0} c(k, l) f_l^\phi \right) f_k^\phi \\
&= 0,
\end{aligned}$$

where in the second equality and the last equality we use (33) and in the third equality we use (29). Therefore, under condition (29) we have an explicit stationary distribution for each value $\rho = m_1(0)$ of the conserved first moment provided it is not larger than ρ_c , given by f^ϕ with $\phi \in D_\phi$ such that $R(\phi) = \rho$.

4.3 Initial conditions and ergodic behaviour

Consider a fixed initial condition $f(0)$ for the limit equation (11) with finite density $\rho = m_1(0) \in (0, \infty)$. A natural corresponding sequence of initial conditions for the particle system are simply product measures ν^L with marginals $\nu^L[\eta_x = \cdot] = f(0)$, in which case $f^L(0) = f(0)$ for all $L \geq 1$. Another useful choice is a conditional version of these measures with a fixed number of particles

$$\pi_{L,N} = \nu^L \left[\cdot \mid \sum_{x \in \Lambda} \eta_x = N \right] \text{ and } f^L(0) = \pi_{L,N}[\eta_x = \cdot]. \quad (34)$$

If N is chosen to increase with L such that $N/L \rightarrow \rho$, then $f^L(0) \rightarrow f(0)$ as $L \rightarrow \infty$ weakly and in total variation distance. The formulation of our main result requires i.i.d. initial conditions (6), which provides permutation invariance of the dynamics and is otherwise used only in (26). Permutation invariance is also given under the conditional measures (34), and the condition introduces only a small negative correlation between different occupation numbers $\eta_x(0)$ and $\eta_y(0)$ of order $1/L$. This leads to a vanishing correction in (26), and the proof can be easily adapted to also cover initial conditions with a fixed number of particles.

Another generic initial condition of the form (34) is to simply distribute N particles uniformly

at random, leading to binomial marginals

$$\binom{N}{k} \left(\frac{1}{L}\right)^k \left(1 - \frac{1}{L}\right)^{N-k} = f_k^L(0) \rightarrow f_k(0) = \frac{\rho^k}{k!} e^{-\rho}, \quad (35)$$

converging to $\text{Poi}(\rho)$ variables as $N, L \rightarrow \infty$ and $N/L \rightarrow \rho$.

Given a family of stationary measures f^ϕ , a natural question is that of ergodicity, i.e. for initial conditions $f(0)$ with first moment $\rho = m_1(0) < \infty$, does $f(t)$ converge to f^ϕ with $R(\phi) = \rho$? While contraction arguments may be possible for particular jump rates $c(k, l)$, we are not aware of general results on convergence to stationary solutions for non-linear dynamical systems that would answer this question. On the restricted state space $\{\eta \in \Omega : \sum_{x \in \Lambda} \eta_x = N\}$ the process $(\eta(t) : t \geq 0)$ is a finite state, irreducible Markov chain, which converges to its unique stationary distribution $\pi_{L,N}$. The equivalence of ensembles for such models has been established (see e.g. [12] and references therein) and ensures weak convergence

$$\pi_{L,N}[\eta_x = k] \rightarrow f_k^\phi \quad \text{as } L, N \rightarrow \infty, N/L \rightarrow \rho \quad (36)$$

provided that $R(\phi) = \rho \leq \rho_c$. For condensing models with $\rho > \rho_c$, the above holds with $\phi = \phi_c < \infty$, which corresponds to a loss of mass in the condensate since the limit has only first moment $\rho_c < \rho$. The sequence of marginals $\pi_{L,N}[\eta_x = \cdot]$ is uniformly integrable if and only if $\rho \leq \rho_c$, in which case convergence in (36) holds also in L^1 . Due to ergodicity for a finite state Markov chain, we have

$$f^L(t) \rightarrow \pi_{L,N}[\eta_x = \cdot] \quad \text{as } t \rightarrow \infty$$

for each finite L , which holds in total variation or L^2 distance.

If the convergence $f^L(t) \rightarrow f(t)$ was uniformly in $t > 0$, this could be used to establish ergodicity for the limit process. But the error bounds arising from Lemma 1 are in fact of order e^{Ct}/L for some $C > 0$, since the branching processes (19) in our coupling argument grow exponentially in time. They are clearly only useful for $t \ll \log L$ (in particular for all fixed $t > 0$), and our proof does not provide uniform convergence. In fact, ergodicity breaking is a well-known phenomenon in the presence of phase transitions, e.g. for the contact process uniqueness of the stationary distribution is lost in infinite volume. For solutions to (11), however, we still expect ergodicity at least for $\rho \leq \rho_c$, and explicit heuristic scaling solutions for particular systems discussed in the next section support this even for $\rho > \rho_c$.

Note that our main result in Theorem 1 holds independently of condition (29) and instead requires boundedness of the rates c . Without condition (29) we still expect a continuous family of stationary distributions for the birth death chain indexed by the first moment with similar ergodicity properties, but we are not aware of related results. Results on some particular cases of non-linear birth death chains can be found in [9].

5 Examples of condensing particle systems

We discuss two examples of processes of type (1) that exhibit condensation and have attracted significant recent research interest. The second has unbounded rates and is not covered by our main theorem, but we include it to illustrate the possible irregular behaviour and non-existence of solutions to (11) related to gelation in growth/aggregation models.

5.1 Zero-range processes

For zero-range processes (ZRP) the jump rates depend only on the occupation of the departure site, and we use the notation

$$c(k, l) = g(k) \quad \text{with } g : \mathbb{N} \rightarrow [0, \infty) \text{ such that } g(k) = 0 \Leftrightarrow k = 0.$$

This leads to the rate equation (11) taking the form

$$\frac{df_k(t)}{dt} = g(k+1)f_{k+1}(t) + \bar{g}(t)f_{k-1}(t) - (g(k) + \bar{g}(t))f_k(t), \quad (37)$$

valid for all $k \geq 0$ with the convention $f_{-1}(t) \equiv 0$. As before this is the master equation of a birth death chain with k -independent birth rate $\bar{g}(t) = \sum_k g(k)f_k(t)$ and time-independent death rate $g(k)$, which have been studied in [35]. ZRPs satisfy (29) for all choices of rates g and exhibit stationary product measures of the form (30).

An interesting example is given by the bounded jump rates

$$g(k) = \begin{cases} 0 & \text{if } k = 0, \\ 1 + \frac{b}{k^\gamma} & \text{if } k \geq 1, \end{cases} \quad (38)$$

with parameters $b > 0$ and $\gamma \in (0, 1]$. For the measures (30) we have $\phi_c = 1$ and stationary weights

$$\begin{aligned} w(n) &= \prod_{k=1}^n \frac{k}{k+b} \sim n^{-b} & \text{for } \gamma = 1, \\ w(n) &= \prod_{k=1}^n \frac{k^\gamma}{k^\gamma + b} \sim \exp\left(-\frac{C}{1-\gamma}n^{1-\gamma}\right), \quad C > 0 & \text{for } \gamma \in (0, 1). \end{aligned}$$

The symbol \sim indicates asymptotic proportionality as $n \rightarrow \infty$, with a power law and a stretched exponential decay, respectively. These models have been studied in great detail (see e.g. [1, 20, 21, 22]), and we have $\rho_c < \infty$ when $\gamma \in (0, 1)$ or $\gamma = 1$ and $b > 2$. If the density $\rho > \rho_c$ the system exhibits condensation, where a finite fraction of all particles concentrates in a single condensate site. Accordingly, f^1 is the stationary measure with maximal density ρ_c of the birth death chain with master equation (37). Intuitively, the dynamic mechanism of condensation in this model is due to the decreasing jump rates $g(k)$ leading to an effective attraction between particles on sites with a large occupation number. The system exhibits an interesting coarsening phenomenon,

where over time the condensed phase concentrates on a decreasing number of lattice sites with increasing occupation numbers. There are only partial rigorous results so far on this question [26], and it has been studied heuristically in [1, 28] and also [29] in terms of scaling solutions of (37). While for initial conditions with $\rho = m_1(0) \leq \rho_c$ ergodicity as discussed in Section 4.3 is expected to apply, for $\rho > \rho_c$ the solution to (37) phase separates for large times according to the scaling ansatz

$$f_k(t) = \underbrace{f_k(t) \mathbb{I}_{[0,1/\sqrt{\epsilon_t}]}(k)}_{:=f_k^{\text{bulk}}(t)} + \underbrace{f_k(t) \mathbb{I}_{(1/\sqrt{\epsilon_t},\infty)}(k)}_{:=f_k^{\text{cond}}(t)}, \quad (39)$$

with a scaling parameter $\epsilon_t \rightarrow 0$ explained below. The bulk part of the distribution applies to finite (fixed) occupation numbers and converges as

$$f_k^{\text{bulk}}(t) \rightarrow f^1 \quad \text{as } t \rightarrow \infty \quad \text{with} \quad \sum_{k \geq 0} k f_k^1 = \rho_c \quad (40)$$

in analogy with the discussion in Section 4.3. The condensed part $f_k^{\text{cond}}(t)$ vanishes pointwise as $t \rightarrow \infty$, taking the scaling form

$$f_k^{\text{cond}}(t) \simeq \epsilon_t^2 h(u), \quad \text{with } u = k\epsilon_t \text{ and } \epsilon_t = t^{-\frac{1}{\gamma+1}}. \quad (41)$$

For $\gamma = 1$, the scaling function h satisfies the second-order linear differential equation

$$h''(u) + \left(\frac{1}{2}u - A + \frac{b}{u} \right) h'(u) + \left(1 - \frac{b}{u^2} \right) h(u) = 0 \quad (42)$$

with additional constraint $h(u) \rightarrow 0$ as $u \rightarrow \infty$ and normalization

$$\int_0^\infty u h(u) du = \sum_{k \geq 1/\sqrt{\epsilon_t}} k f_k^{\text{cond}}(t) = \rho - \rho_c. \quad (43)$$

The solutions have been discussed in detail in [1], and show a unimodal bump corresponding to the mass distribution in the condensed phase with a total density of $\rho - \rho_c$, equaling the excess mass which is missing in the bulk part. For $\gamma < 1$, the analogue to (42) is more complicated and a detailed analysis is provided [28]. With the first moment being conserved, the simplest characterization of condensation dynamics is given by the second moment of the occupation numbers, $m_2(t)$. Using the scaling ansatz (39), (41) and computing $\mathcal{L}_{BD} k^2$, and with (40) and (43) this is dominated by the condensed part and diverges as a power law

$$m_2(t) = \sum_{k \geq 0} k^2 f_k(t) \sim \epsilon_t^{-1} = t^{\frac{1}{\gamma+1}} \quad \text{as } t \rightarrow \infty.$$

5.2 Explosive condensation processes and gelation

Explosive condensation processes (ECP) have been introduced in [17] and further studied in [3, 18] on a heuristic level. The jump rates are of the form

$$c(k, l) = k^\lambda(d + l^\lambda) \quad \text{with parameters } \lambda \geq 1 \text{ and } d > 0, \quad (44)$$

are unbounded and diverge super-linearly with occupation numbers on departure and target site. For $\lambda = 1$ this model is called the inclusion process which has been studied on a rigorous level in [36, 15]. While our result does not apply to unbounded rates, (11) still represents the only possible limit dynamics for f_k , and we expect convergence to actually hold at least as long as it has a unique solution. Rates of the form (44) are related to collision kernels in aggregation models which have attracted significant research interest (see e.g. [18] and references therein).

The rates (44) satisfy condition (29) and we have product measures of the form (30) with $\phi_c = 1$ and

$$\begin{aligned} w(n) &= \frac{\Gamma(d+n)}{n!\Gamma(d)} \sim n^{d-1} && \text{for } \lambda = 1, \\ w(n) &= \prod_{k=1}^n \frac{(k-1)^\lambda + d}{k^\lambda} \sim n^{-\lambda} && \text{for } \lambda > 1. \end{aligned} \quad (45)$$

Therefore, $\rho_c < \infty$ for $\lambda > 2$ and as for models with bounded rates we expect $f(t) \rightarrow f^\phi$ as $t \rightarrow \infty$ for all initial conditions with $m_1(0) = \rho \leq \rho_c$. If $\rho > \rho_c$, we expect a scaling solution in analogy to ZRPs.

The exchange-driven growth model studied in [5] corresponds to rates (44) in the degenerate case $d = 0$, and provides a detailed analysis of the condensed part of the scaling solution. Note that in this case $w(n) = \delta_{0,n}$ and the mean-field equation has an absorbing state corresponding to $f_k = \delta_{0,k}$ as the only stationary distribution for all $\lambda > 0$, effectively setting $\rho_c = 0$. Still, $m_1(t)$ is conserved and the dynamics of the particle system is not irreducible, more and more sites become emptied over time and cannot get occupied again thereafter. The limiting master equation (11) can be written as

$$\frac{d}{dt} f_k(t) = m_\lambda(t) [(k+1)^\lambda f_{k+1}(t) + (k-1)^\lambda f_{k-1}(t) - 2k^\lambda f_k(t)], \quad (46)$$

for all $k \geq 0$, using $f_{-1}(t) \equiv 0$. This involves the moment $m_\lambda(t)$ which can be absorbed in a time change $\tau_t = \int_0^t dt' m_\lambda(t')$, leading to a standard birth death chain with symmetric rates k^λ . Since $\rho_c = 0$ all initial conditions with $\rho = m_1(0) > 0$ lead to phase separated solutions of the form (39)

$$f_k(t) = f_k^{\text{bulk}}(t) + f_k^{\text{cond}}(t), \quad (47)$$

now with $f_k^{\text{bulk}}(t) \rightarrow \delta_{k,0}$. The results reported in [5] refer to f_k^{cond} , which for $\lambda < 2$ again

exhibits a scaling form

$$f_k^{\text{cond}}(t) = \epsilon_t^2 h(u), \quad \text{with } u = k\epsilon_t, \quad \epsilon_t = \tau_t^{-\alpha} \text{ and } \alpha = \frac{1}{2-\lambda}. \quad (48)$$

The scaling function again satisfies a second-order linear differential equation

$$(2-\lambda) \frac{d^2}{du^2} (u^\lambda h(u)) + u \frac{d}{du} h(u) + 2h(u) = 0, \quad (49)$$

subject to normalization, which has an explicit solution

$$h(u) = \frac{(2-\lambda)^{2/(2-\lambda)}}{\Gamma(1/2-\lambda)} u^{1-\lambda} \exp\left(-\frac{u^{2-\lambda}}{(2-\lambda)^2}\right). \quad (50)$$

For $\lambda > 2$ there is no solution to the limit dynamics (11), which exhibit instantaneous blow up of second moments – also called gelation in the context of aggregation models (see e.g. [7]). On the level of the particle system this corresponds to the explosive condensation phenomenon studied in [17, 3, 18] for $d > 0$, where the time to reach the condensed state vanishes with increasing system size even in one-dimensional geometries. On the complete graph with $d = 0$ the behaviour can again be characterized through the second moment as reported in [5]

$$m_2(t) \sim \begin{cases} t^\beta & , \lambda < 3/2 \\ \exp(Ct) & , \lambda = 3/2 \text{ for some } C > 0 \\ (t_c - t)^\beta & , 3/2 < \lambda < 2 \text{ for some } t_c > 0 \\ \infty & , \lambda > 2 \end{cases}. \quad (51)$$

The dynamical exponent for the power law cases above is given by $\beta = (3 - 2\lambda)^{-1}$, and for $\lambda > 3/2$ the system exhibits finite-time blow up at t_c , which becomes instantaneous for $\lambda > 2$. The boundary case $\lambda = 2$ shows interesting multiscaling behaviour as discussed in Section 3B [5]. Note that for $d > 0$ with (45) only $\lambda > 2$ leads to $\rho_c < \infty$ and condensation is always explosive as mentioned above.

6 Discussion

We have established the mean-field equation (11) as the limit dynamics of stochastic particle systems, which provides an important ingredient for a rigorous analysis of the coarsening dynamics of condensing stochastic particle systems. Our result holds under arguably quite restrictive conditions, which we discuss in detail in the following.

- Theorem 1 is formulated for i.i.d. initial conditions (6), and we have discussed in Section 4 how this can be extended to conditional product measures which introduce vanishing correlations and are permutation invariant. In our proof, permutation invariance is only used to establish existence in Section 3.1. This makes use of (8) implying that the single site process $\eta_x(t)$ provides a realization of the limiting birth death chain. Since all estimates in

Section 3.1 hold uniformly in x , a similar argument can be used to establish tightness for the empirical process $(F_k(\eta_t) : k \geq 0)$. This would allow for non-permutation invariant initial conditions with vanishing correlations and a result on convergence of $f_k^L(t)$.

- Mean-field equations (11) are often used as approximations in other geometries such as symmetric or asymmetric dynamics on d -dimensional regular lattices. As usual, the larger the dimension the better the approximation, see e.g. [2, 18, 3] for details. Since our result does not involve any time scaling, mean-field averaging of the birth and death rates (28) is achieved by a diverging number of neighbours of each lattice site. This is a crucial ingredient in our proof in (23) and in fact essential for any rigorous derivation of (11). Our arguments could be directly extended to graphs which are not complete but have a version of the above property.
- Condensing stochastic particle systems exhibit several time scales diverging with the system size (e.g. for ZRPs, this has been studied in [2]), some of which have been identified recently also on a rigorous basis including hydrodynamics [27] and also metastable dynamics of the condensate [23, 24]. As we discussed in Section 4 convergence in our result does not hold uniformly in time, and error estimates vanish on time scales at most of order $\log L$ due to the coupling with branching processes.
- Boundedness of jump rates (9) is the most restrictive condition that we expect to be not necessary for the limit result to hold, but which would require a significant extension of our proof including e.g. a priori bounds on occupation numbers to use cut-off arguments. For the inclusion process (44) with $\lambda = 1$ our result can be established with other techniques, which is current work in progress. However, the example of explosive processes with $\lambda > 2$ shows that some growth conditions on the rates are necessary for convergence to (11) to hold. In cases of instantaneous blow up, the single site process $\eta_x(t)$ does not have well-defined limit dynamics for any $t > 0$.

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